Least-action Principle Applied to the Kepler Problem

S. Olszewski

Institute of Physical Chemistry of the Polish Academy of Sciences,
Kasprzaka 44/52, 01-224 Warsaw, Poland

Reprint requests to Prof. S. O.; e-mail: olsz@ichf.edu.pl

Z. Naturforsch. **59a,** 375 – 381 (2004); received March 1, 2004

The least-action principle is examined for the two-body Kepler problem. This examination allows one to couple the eccentricity parameter of the Kepler orbit with the size of the major semiaxis of that orbit. The obtained formula is applied to an estimate of eccentricities characteristic for a set of the planetary and satellitary tracks.

Key words: Eccentricity of the Kepler Orbits; Least-action Principle.

1. Introduction: Dependence of the Eccentricity *e* on the Major Semiaxis a_0 of the Kepler Orbit

The Kepler problem is one of the best known in classical mechanics. Being based partly on observations, it accounts of the motion of a planet, or satellite, along an elliptical trajectory extended about a gravitational center. The period T of the motion and the major semiaxis a_0 are coupled by the well-known Kepler law

$$GM_S T^2 = 4\pi^2 a_0^3, \quad (1)$$

where G is the gravitational constant, M_S is the mass of the Sun in the planet case, or the mass of the planet in the satellite case. In (1) it is assumed – for the sake of simplicity – that the mass M_S remains at rest. Simultaneously, the eccentricity e defining the elliptical trajectory of a planet, or satellite, is considered as independent of a_0 or T [1 – 6].

In spite of the fundamental importance of the Kepler problem its mechanics seems to be discussed insufficiently. Here we have in mind the least-action principle applied to a conservative system. The principle makes reference to the variational foundations of mechanics and is based on the idea of the action function. In practice, the action variables $J_1, J_2, \ldots$, considered for the Kepler problem, are parameters conjugated to the corresponding angle variables of the moving body [7, 8]. Both kinds of the action and angle variables serve to transform the Hamiltonian into a function dependent solely on $J_1, J_2, \ldots$. This technique is useful because it enables one to calculate the frequencies of the periodic motion without finding a complete solution of the motion of the system. In the next step, the same action variables $J_1, J_2, \ldots$ occur useful in the quantization process which takes an easy form according to the well-known Bohr-Sommerfeld rules [9, 10].

But beyond of $J_1, J_2, \ldots$ an elementary action function S_0 can be defined, equal to double the kinetic energy of a body integrated over a time interval passed by that body along its trajectory. The least-action principle states that – for a conservative system – this action function should be at minimum [8, 11]. In the next step, the derivative of S_0 calculated with respect to the energy E_p of the body at some time t provides us with the time interval of the motion of the body, beginning from some time t_0:

$$\frac{\partial S_0}{\partial E_p} = t - t_0. \quad (2)$$

The aim of the present paper is to point out that (2) can be satisfied only on the condition that the eccentricity e of the planetary, or satellitary, track depends on the major semiaxis a_0 of the Kepler orbit.

The Cartesian coordinates of a body on the Kepler orbit are

$$x = a_0(\cos u - e), \quad (3)$$

$$y = a_0(1 - e^2)^{1/2} \sin u, \quad (3a)$$

where the coordinate u depends on the time t by the equation [8]

$$\frac{2\pi}{T} t \equiv \omega t = u - e \sin u. \quad (3b)$$

Here we assumed that the beginning of the motion is at the perihelion of the Kepler orbit [8]. Obviously, $t = 0$ at $u = 0$; the frequency of the motion about the gravitation center has been put equal to ω.

For a moment let us assume that e is constant in a way similar to a_0. Because of (3), (3a) and (3b) we have

$$\begin{aligned} \dot{x}^2+\dot{y}^2 &= a_0^2\left[\sin^2 u+(1-e^2)\cos^2 u\right]\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)^2 \\ &= a_0^2(1-e^2\cos^2 u)\omega^2\frac{1}{(1-e^2\cos u)^2} \qquad (4) \\ &= a_0^2\omega^2\frac{1+e\cos u}{1-e\cos u}. \end{aligned}$$

The integral of the expression (4), extended over an interval of t beginning with $t = t_1 = 0$ and ending at some $t = t_2$, gives

$$\begin{aligned} \int_0^{t_2}(\dot{x}^2+\dot{y}^2)\mathrm{d}t &= a_0^2\omega^2\int_0^{t_2}\frac{1+e\cos u}{1-e\cos u}\mathrm{d}t \\ &= a_0^2\omega\int_0^{u_2}(1+e\cos u)\mathrm{d}u \qquad (5) \\ &= a_0^2\omega(u+e\sin u)\Big|_{u=0}^{u=u_2}, \end{aligned}$$

since $u(t_1) = u(0) = 0$ and

$$\frac{\mathrm{d}u}{\mathrm{d}t}=\frac{\omega}{1-e\cos u}; \qquad (6)$$

see (3b). The integral giving the elementary action S_0, equal to the integral of the double kinetic energy extended along the time interval $(0,t_2)$, is

$$\begin{aligned} S_0 &= m_\mathrm{p}\int_0^{t_2}(\dot{x}^2+\dot{y}^2)\mathrm{d}t \\ &= m_\mathrm{p}a_0^2\omega(u+e\sin u)\Big|_{u=0}^{u=u_2}, \end{aligned} \qquad (7)$$

where m_p is the mass of a planet or satellite.

The fundamental equation (2), valid for any planet or satellite trajectory, becomes therefore

$$\begin{aligned} t_2 &= \frac{\partial S_0}{\partial E_\mathrm{p}} = \frac{\dfrac{\partial S_0}{\partial a_0}}{\dfrac{\partial E_\mathrm{p}}{\partial a_0}} \\ &= \frac{\left(2a_0\omega+a_0^2\dfrac{\partial\omega}{\partial a_0}\right)(u_2+e\sin u_2)}{\dfrac{GM_\mathrm{S}}{2a_0^2}} \qquad (8) \\ &= \frac{a_0^3\omega(u_2+e\sin u_2)}{GM_\mathrm{S}} = \frac{1}{\omega}(u_2+e\sin u_2). \end{aligned}$$

In the calculation of (8) we took into account

$$E_\mathrm{p} = -\frac{GM_\mathrm{S}m_\mathrm{p}}{2a_0}, \qquad (9)$$

which is the energy of the planet or satellite, as well as the relations (1) and (3b). Since from the third Kepler law given in (1) we have

$$\frac{\partial T}{\partial a_0}=\frac{3}{2}\frac{T}{a_0}, \qquad (10)$$

the derivative of the expression $a_0^2\omega$ entering (7) calculated with respect to a_0 gives

$$\begin{aligned} \frac{\partial(a_0^2\omega)}{\partial a_0} &= 2a_0\omega+a_0^2\frac{\partial\omega}{\partial a_0} \\ &= 2a_0\omega+2\pi a_0^2\left(-\frac{1}{T^2}\right)\frac{3}{2}\frac{T}{a_0} \qquad (11) \\ &= \frac{1}{2}a_0\omega. \end{aligned}$$

Our task is now to compare the result of (8) with $t_2 = t_2(u_2)$ calculated from (3b) for an arbitrary fraction of the period T of rotation of the celestial body about the gravitational center. This is done for different values of

$$t_2 = \lambda T, \qquad (12)$$

where λ is a constant: $0 < \lambda \le 1$. The mentioned comparison is presented in Table 1, where – for the sake of convenience – the variable $u = u_2$ has been taken as a parameter. It is evident that solely at $\lambda = \frac{1}{2}$ and $\lambda = 1$ there exists an agreement between $t_2 = t_2(u_2)$ taken from (3b) and t_2 calculated from (8); the case $\lambda = 0$ has been omitted as being a trivial one.

A discrepancy obtained between t_2 in (3b) and t_2 given by Eq. (8), illustrated by the second and the third

Table 1. A comparison of the time interval λT of the motion of a celestial body performed along the Kepler orbit with the time interval calculated from the elementary action function; see (3b), (8) and (12). The orbit eccentricity e is assumed equal to a constant number. T is the circulation period of the body about the gravitational center. The beginning point of the motion is assumed in the perihelion when the position of the center of force is assumed in the Sun.

u_2 see (3b) for a reference between t_2 and u_2	λ obtained from t_2 calculated according to (3b)	λ obtained from t_2 calculated in (8)
$\frac{1}{4}\pi$	$\frac{1}{8}-\frac{\sqrt{2}}{4\pi}e$	$\frac{1}{8}+\frac{\sqrt{2}}{4\pi}e$
$\frac{1}{2}\pi$	$\frac{1}{4}-\frac{1}{2\pi}e$	$\frac{1}{4}+\frac{1}{2\pi}e$
$\frac{3}{4}\pi$	$\frac{3}{8}-\frac{\sqrt{2}}{4\pi}e$	$\frac{3}{8}+\frac{\sqrt{2}}{4\pi}e$
π	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{5}{4}\pi$	$\frac{5}{8}+\frac{\sqrt{2}}{4\pi}e$	$\frac{5}{8}-\frac{\sqrt{2}}{4\pi}e$
$\frac{3}{2}\pi$	$\frac{3}{4}+\frac{1}{2\pi}e$	$\frac{3}{4}-\frac{1}{2\pi}e$
$\frac{7}{4}\pi$	$\frac{7}{8}+\frac{\sqrt{2}}{4\pi}e$	$\frac{7}{8}-\frac{\sqrt{2}}{4\pi}e$
2π	1	1

column of Table 1, implies that e entering S_0 should depend on a_0:

$$e = e(a_0). \tag{13}$$

2. Equation for $e(a_0)$

This dependence can be examined when the Fourier expansion for u is applied [6, 12, 13]:

$$u = \omega t + 2\sum_{n=1}^{\infty}\frac{1}{n}J_n(ne)\sin(n\omega t). \tag{14}$$

Here J_n is the nth-order Bessel function of the first kind. Because e is a small number, the expansion (14) – as well as the power expansions for J_n – can be limited to either one or two first terms. By substituting the expansion (14) for $u = u_2$ into the derivative of S_0, we obtain

$$\frac{\partial\left(\frac{S_0}{m_\mathrm{p}}\right)}{\partial a_0} \cong \frac{1}{2}a_0\omega[\omega t_2 + e' + e\sin(\omega t_2 + e')] \tag{15}$$

$$+a_0^2\omega\left[\frac{\partial e'}{\partial a_0} + \frac{\partial e}{\partial a_0}\sin(\omega t_2 + e') + e\cos(\omega t_2 + e')\frac{\partial e'}{\partial a_0}\right],$$

where

$$e' \approx e\sin(\omega t_2). \tag{15a}$$

In the next step, because of the relation

$$t_2\frac{\partial\left(\frac{E_\mathrm{p}}{m_\mathrm{p}}\right)}{\partial a_0} = t_2\frac{GM_\mathrm{S}}{2a_0^2} = t_2\frac{2\pi^2 a_0}{T^2} = \frac{1}{2}a_0\omega^2 t_2, \tag{16}$$

the expression (15), substituted into the first part of (8), gives

$$\begin{aligned} 0 &= \frac{1}{2}a_0\omega e' + \frac{1}{2}a_0\omega y + a_0^2\omega\frac{\partial e'}{\partial a_0} + a_0^2\omega\frac{\partial y}{\partial a_0} \\ &= \frac{1}{2}a_0\omega(e'+y) + a_0^2\omega\frac{\partial}{\partial a_0}(e'+y), \end{aligned} \tag{17}$$

or

$$e' + y = -2a_0\frac{\partial}{\partial a_0}(e'+y), \tag{17a}$$

where

$$y = e\sin(\omega t_2 + e'). \tag{18}$$

The solution of (17a) is

$$\ln\left[a_0^{1/2}(e'+y)\right] = \text{const}' \tag{19}$$

or

$$a_0^{1/2}\left[e' + e\sin(\omega t_2 + e')\right] = \text{const.} \tag{19a}$$

It is evident from (19a) that e depends both on a_0 and t_2. For example an explicit differentiation of (19a) with respect to a_0 gives successively the equations

$$ea_0^{-1/2} + 2a_0^{1/2}\frac{\partial e}{\partial a_0} = 0 \tag{20}$$

for $\omega t_2 = \frac{\pi}{2}$ and $\omega t_2 = \frac{3}{2}\pi$,

$$\frac{1}{2}ea_0^{-1/2}\left(\sqrt{2} + \frac{1}{2}e\right) + a_0^{1/2}\left(\sqrt{2} + e\right)\frac{\partial e}{\partial a_0} = 0 \tag{20a}$$

for $\omega t_2 = \frac{\pi}{4}$ and $\omega t_2 = \frac{7}{4}\pi$, and

$$\frac{1}{2}ea_0^{-1/2}\left(\sqrt{2} - \frac{1}{2}e\right) + a_0^{1/2}\left(\sqrt{2} - e\right)\frac{\partial e}{\partial a_0} = 0 \tag{20b}$$

for $\omega t_2 = \frac{3}{4}\pi$ and $\omega t_2 = \frac{5}{4}\pi$. If the terms $\frac{1}{2}e$ and e entering the brackets in (20a) and (20b) are omitted, these equations are reduced to the same differential equation for $e(a_0)$, which is given in (20). For $\omega t_2 = \pi$ and

$\omega t_2 = 2\pi$ Eq. (20) is fulfilled identically. A solution of (20) is

$$a_0^{1/2} e = \text{ const} \tag{21}$$

which is a result proportional to an average of (19a) calculated over $\frac{T}{2}$. Evidently, the average of (19a) calculated over the whole time period T is equal to zero. Equation (21) implies that the eccentricities e_1 and e_2 of two non-interacting bodies 1 and 2 circulating about a common gravitational center are referred to the major semiaxes a_{01} and a_{02} of the bodies trajectories by the equation

$$\frac{e_1}{e_2} = \frac{a_{02}^{1/2}}{a_{01}^{1/2}}. \tag{21a}$$

The results of (21) and (21a) are used in Section 4.

3. Least-action Principle for the Kepler Problem, Examined in the Framework of the Euler-Jacobi Formulation

Equation (20) can be obtained also from a direct application of the so-called Euler-Jacobi formulation of the least-action principle [14–16]. This formulation had as its main idea, to transform the time t into some new variable

$$\tau = \tau(t) \tag{22}$$

and perform the variational calculation with the aid of τ instead of t. For the case of the Kepler problem a natural substitution for τ is

$$u = u(t), \tag{22a}$$

where u is the eccentric anomaly introduced in Sect. 2 and coupled with t by the formula (3b). In effect, for the problem of a planar motion we obtain the requirement [15, 16]

$$\delta \int \sqrt{E_\mathrm{p} - V} \sqrt{m_\mathrm{p} \left[\left(\frac{\mathrm{d}x}{\mathrm{d}u} \right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}u} \right)^2 \right]} \mathrm{d}u = 0, \tag{23}$$

where E_p is the total energy (9) and

$$V = -\frac{GM_\mathrm{S} m_\mathrm{p}}{(x^2+y^2)^{1/2}} \tag{9a}$$

is the potential energy of the moving body. Because of (3) and (3a) we have

$$x^2 + y^2 = a_0^2 (1 - e\cos u)^2, \tag{24}$$

and in view of (3) and (3a)

$$\begin{aligned} \left(\frac{\mathrm{d}x}{\mathrm{d}u} \right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}u} \right)^2 &= a_0^2 [\sin^2 u + (1-e^2)\cos^2 u] \\ &= a_0^2 (1 - e^2 \cos^2 u), \end{aligned} \tag{24a}$$

so the expression submitted to variation in (23) becomes:

$$\begin{aligned} &\left[E_\mathrm{p} + \frac{GM_\mathrm{S} m_\mathrm{p}}{(1-e\cos u) a_0} \right]^{1/2} \left[m a_0^2 (1 - e^2 \cos^2 u) \right]^{1/2} \\ &= \left[E_\mathrm{p} m_\mathrm{p} a_0^2 (1 - e^2\cos^2 u) + \frac{GM_\mathrm{S} m_\mathrm{p}^2 a_0^2 (1 - e^2 \cos^2 u)}{(1 - e\cos u) a_0} \right]^{1/2} \\ &= \left[E_\mathrm{p} m_\mathrm{p} a_0^2 + GM_\mathrm{S} m_\mathrm{p}^2 a_0 + GM_\mathrm{S} m_\mathrm{p}^2 a_0 e \cos u - E_\mathrm{p} m_\mathrm{p} a_0^2 e^2 \cos^2 u \right]^{1/2} \\ &= \left(\frac{1}{2} GM_\mathrm{S} m_\mathrm{p}^2 a_0 \right)^{1/2} \left(1 + 2e\cos u + e^2 \cos^2 u \right)^{1/2} = \left(\frac{1}{2} GM_\mathrm{p} m_\mathrm{p}^2 a_0 \right)^{1/2} (1 + e\cos u). \end{aligned} \tag{25}$$

In the last but one transformation in (25) we took into account the formula (9). Due to (25), the variational problem of (23) becomes

$$\delta \left(\frac{1}{2} GM_\mathrm{S} m_\mathrm{p}^2 a_0 \right)^{1/2} \int (1 + e\cos u) \mathrm{d}u \tag{26}$$

$$= \delta \left(\frac{1}{2} GM_\mathrm{S} m_\mathrm{p}^2 a_0 \right)^{1/2} \left[\int \mathrm{d}u + e \int \cos u \mathrm{d}u \right] = 0.$$

A full gyration of a body about a gravitational center provides us with the interval

$$0 \le u \le 2\pi. \tag{27}$$

In this case the e-dependent component of the integral entering (26) vanishes. Consequently, by allowing for variation of a_0, we obtain from (26) and (27) that

$$\delta\left(\frac{1}{2}GM_\mathrm{S}m_\mathrm{p}^2a_0\right)^{1/2}2\pi = \left(\frac{1}{2}GM_\mathrm{S}m_\mathrm{p}^2\right)^{1/2}\pi a_0^{-1/2}\delta a_0 = 0 \tag{28}$$

or

$$\delta a_0 = 0. \tag{28a}$$

In view of (28a) the term equal to the first component of (26) can be considered as a constant. We assume that solely the second component of the expression (26) may provide us with a coupling between e and a_0. By neglecting the constant factor of $\left(\frac{1}{2}GM_\mathrm{S}m_\mathrm{p}^2\right)^{1/2}$ we obtain from the second integral in (26) between the limits u_0 and u_1:

$$\delta(a_0^{1/2}e)(\sin u_1 - \sin u_0) = 0. \tag{29}$$

This gives us a dependence between e and a_0 because the sin-difference in (29) can be considered as a multiplier which does not influence the variational process. In effect, from (29) we have

$$\frac{1}{2}a_0^{-1/2}\delta a_0 e + a_0^{1/2}\delta e = 0, \tag{29a}$$

which – when divided by $\frac{1}{2}\delta a_0$ – gives the equation equivalent to (20). The same result can be obtained by applying the variational procedure to the original expression of the kinetic energy [8, 11, 13]:

$$\delta\int m_\mathrm{p}\left[\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2\right]\mathrm{d}t = 0, \tag{30}$$

called the Euler-Maupertuis formulation of the least-action principle.

4. Calculation of Eccentricities for the Planet and Satellite Trajectories. Discussion

Equation (21) or (21a) can be readily applied to the calculation of e of the Kepler tracks. From the nine planets arranged according to their increasing distance from the Sun, the planet Jupiter is a central one; this is also the heaviest planet of the system [17]. Putting

$$a_0^{\mathrm{Jupiter}} \equiv x^{\mathrm{Jupiter}} = 1 \tag{31}$$

Table 2. Theoretical eccentricities e^{theor} of the planet trajectories calculated on the basis of (21a) compared with the observed eccentricities e^{obs}; see [17]. The observed eccentricity of the Jupiter trajectory has been taken as the basic parameter in order to define e^{theor} of the orbits of the remaining planets. The major semiaxis $a_0 \equiv x$ of the planet trajectories are also listed with the x value of Jupiter put equal to 1.

Planet	x	e^{theor}	e^{obs} see [17]	Error of e^{theor} see (32)
Mercury	0.07440	0.176	0.2056	0.17
Venus	0.13903	0.129	0.0068	17.97
Earth	0.19220	0.109	0.0167	5.53
Mars	0.29286	0.089	0.0933	0.05
Jupiter	1.00000	–	0.0480	–
Saturn	1.83338	0.035	0.0560	0.60
Uranus	3.68864	0.025	0.0460	0.84
Neptune	5.77782	0.020	0.0100	1.00
Pluto	7.59759	0.017	0.2480	13.60

Table 3. Orbits of the satellites of Jupiter and their eccentricities. The observed eccentricity (e^{obs}) of Elara has been taken as a known parameter in the calculation of e^{theor} of the remaining satellite orbits; (21a). The major semiaxis $x = a_0$ of the orbit of Elara is taken as a unit distance.

Satellite	x	e^{theor}	e^{obs} see [17]	Error of e^{theor} see (32)
Callisto	0.160	0.517	0.010	50.70
Leda	0.946	0.213	0.146	0.46
Himalia	0.977	0.209	0.158	0.32
Lysithea	0.977	0.207	0.130	0.59
Elara	1.000	–	0.207	–
Ananke	1.763	0.156	0.170	0.11
Carme	1.904	0.150	0.210	0.40
Pasiphae	1.985	0.147	0.380	1.58
Sinope	2.019	0.146	0.280	0.92

and taking into account that

$$e^{\mathrm{Jupiter}} = 0.048, \tag{31a}$$

the eccentricities e^{theor} of the remainder eight planets of the solar system can be calculated. The results for e^{theor}, together with the observed data e^{obs} for e and the planet distances $a_0 = x$, are listed in Table 2. The error given in Table 2, and subsequent Tables, is defined by the relation

$$\mathrm{error} = \frac{e_>}{e_<} - 1, \tag{32}$$

where $e_>$ is the larger and $e_<$ is the smaller quantity of the pair of e^{obs} and e^{theor} obtained for a given planet (or satellite); see Tables 2 – 5.

The error (32) is below unity for Mars, Saturn and Uranus, which have their orbits closest to that of Jupiter. More distant planet orbits, which are Earth and

Table 4. Orbits of the satellites of Uranus and their eccentricities. The observed eccentricity (e^{obs}) of Umbriel has been taken as a known parameter in the calculation of e^{theor} of the remaining satellite orbits of Uranus; (21a). The major semiaxis $x = a_0$ of the orbit of Umbriel is taken as a unit distance.

Satellite	x	e^{theor}	e^{obs} see [17]	Error of e^{theor} see (32)
Miranda	0.488	0.0050	0.0170	2.40
Ariel	0.718	0.0041	0.0028	0.46
Umbriel	1.000	–	0.0035	–
Titania	1.640	0.0027	0.0024	0.13
Oberon	2.193	0.0024	0.0007	2.43

Table 5. Theoretical eccentricities e^{theor} of the orbits taken as a basis in the calculations presented in Table 2, 3, and 4 (Jupiter, Elara and Umbriel). The $e^{\mathrm{theor}} = e_i$ are calculated from (33) with the aid of the observed data (e_{i-1}, e_{i+1} for the eccentricities and $a_{0,i-1}, a_{0,i+1}$ for the major semiaxes) of the orbits neighboring to i and the major semiaxis $a_{0,i}$ of the orbit i.

Considered celestial body (i)	Bodies on the neighboring orbits ($i-1, i+1$)	e^{theor}	e^{obs} see [17]	Error of e^{theor} calculated from (32)
Jupiter	Mars, Saturn	0.0630	0.0480	0.31
Elara	Lysithea, Ananke	0.1780	0.2070	0.16
Umbriel	Ariel, Titania	0.0024	0.0035	0.46

Neptune, have their error equal to 5.5 and 1.0, respectively. The largest error – much above 10 – is obtained for Venus and Pluto. Rather accidentally, the error of e^{theor} for Mercury is much below 1.0. These data show that for numerous planets the dependence of e on a_0 should be in fact much more complicated than that represented by (21). The complication can be attributed to the many-body interactions between planets. A dependence of e on a_0, different than that given in (21), should apply especially to a group of relatively light and relatively close planet orbits, like those of Mercury, Venus, and Earth.

A comparison of the observed data with the theoretical values obtained for e seems to be more difficult for the satellite orbits than for the planetary ones because numerous eccentricity data for the satellite tracks are either variable, or uncertain, and the influence of the many-body effects exerted on the solution of the two-body Kepler problem is here much intensified. Many planets – especially Jupiter, Saturn, Uranus and Neptune – have a large number of satellites whose a_0 are approximately equal, and the masses of these satellites are often very similar [17]. For example, the largest error of e^{theor} is found in the case of the satellites of Saturn. These satellites are large in their number, are placed on not much distant orbits and have not very much different masses. The assumption of a coplanar position of the orbits having a common gravitational center can be another source of the error.

A much better situation is attained for the satellites of Jupiter, for which the well-established observed eccentricities beginning with Callisto and the next eight satellites – taken successively with an increasing distance from the Jupiter center – are considered; see Table 3. The observed eccentricity of Elara, having a central position of its orbit distance from the Jupiter center, has been taken as a starting point to calculate the orbit eccentricities of the remainder of the Jupiter satellites. A very large error of e^{theor} is found solely for Callisto, but from the seven remaining e^{theor}, six of them (those for Leda, Himalia, Lysithea, Ananke, Carme and Sinope) have their error below 1, and the error of one of e^{theor} (that of the orbit of Pasiphae) is not much above that number.

A similar comparison of the eccentricities is done for the five satellites of Uranus (Miranda, Ariel, Umbriel, Titania, Oberon) for which the observed eccentricity of Umbriel has been chosen to establish the remainder of the satellite eccentricity data; Table 4. The satellites which have their orbits closest to that of Umbriel give the error of e^{theor} much less than 1.0 (Ariel and Titania), whereas the same error for the external satellite orbits, those of Miranda and Oberon, exceeds 2.0.

An estimate of eccentricities of the orbits taken as the basis of calculations done in Table 2, 3, and 4 (Jupiter, Elara and Umbriel, respectively) can be attained by considering the observed data for e and a_0 of the two neighboring orbits of these bodies (Mars and Saturn for Jupiter, Lysithea and Ananke for Elara, Ariel and Titania for Umbriel); see Table 5. The final result for e^{theor} is calculated as the arithmetical mean of e^{theor} obtained from any member of the mentioned orbit pair. Therefore, with the aid of (21a), we have

$$e_i = \frac{1}{2a_{0,i}^{1/2}} \left(e_{i-1} a_{0,i-1}^{1/2} + e_{i+1} a_{0,i+1}^{1/2} \right), \tag{33}$$

where the index i labels the examined orbit, and

$$a_{0,i-1} < a_{0,i} < a_{0,i+1}. \tag{33a}$$

The error of all e^{theor} presented in Table 5 is much below 1.0.

5. Summary

The least-action principle is applied to the motion of a celestial body along the Kepler orbit. This principle – called also the Euler-Maupertuis principle – is only a special case of more general Lagrange or Hamilton principles valid upon the following restrictions [18]: (i) the Lagrangian function should not be explicitly time dependent, so the energy of the system is a conserved quantity both on the actual and the varied paths; (ii) for the varied paths the changes of the body position at the terminal positions should vanish, which – for a closed path – makes this requirement applicable to the beginning-end point of the path; (iii) the potential function entering the Lagrangian should be velocity-independent. With the above restrictions, which can be assumed to be satisfied for a body moving on the Kepler orbit, the Euler-Maupertuis principle becomes equivalent to the Lagrange principle [18].

In Sect. 1 we demonstrated that the Sommerfeld-Landau formula for the derivative of the action function with respect to energy cannot be satisfied unless the eccentricity of the orbit is a function of the major orbital semiaxis. The corresponding equation, coupling the eccentricities and major semiaxes of the orbits having a common gravitational center, is derived in Section 2. The same equation can be derived by a direct application of the least-action principle in both the Euler-Jacobi and conventional (Maupertuis) variational formulations; Section 3. The results of the equation which couples e and a_0 are discussed in comparison with the observed data in Section 4. No influence of the mass ratio of the rotating body to that of the central body on the equation defining the orbit eccentricity is considered.

[1] V. M. Blanco and S. W. McCuskey, Basic Physics of the Solar System, Addison-Wesley, Reading, Massachusetts 1961.
[2] E. Finlay-Freundlich, Celestial Mechanics, Pergamon, New York 1958.
[3] P. M. Fitzpatrick, Principles of Celestial Mechanics, Academic, New York 1970.
[4] S. W. McCuskey, Introduction to Advanced Dynamics, Addison-Wesley, Reading, Massachusetts 1959.
[5] F. R. Moulton, An Introduction to Celestial Mechanics, 2nd ed., Macmillan, New York 1958.
[6] T. E. Sterne, Introduction to Celestial Mechanics, Interscience, New York 1960.
[7] H. Goldstein, Classical Mechanics, 2nd ed., Addison-Wesley, Reading, Massachusetts 1980.
[8] A. Sommerfeld, Mechanik, Akademische Verlagsgesellschaft, 4th ed., Leipzig 1949.
[9] A. Sommerfeld, Atombau und Spektrallinien, Vol. 1, 5th ed., Vieweg, Braunschweig 1931.
[10] J. C. Slater, Quantum Theory of the Atomic Structure, Vol. 1, McGraw-Hill, New York 1960.
[11] L. D. Landau and E. M. Lifshitz, Mechanics, Pergamon Press, Oxford 1960.
[12] H.C. Plummer, Dynamical Astronomy, Cambridge University Press, Cambridge 1918 (reprinted by Dover Publ., New York 1960).
[13] W. Rubinowicz and W. Królikowski, Theoretical Mechanics (in Polish), 7th ed., Wydawnictwo Naukowe PWN, Warszawa 1995.
[14] C. Lanczos, The Variational Principles of Mechanics, 4th ed., Dover Publications, New York 1986.
[15] P. Frank and R. v. Mises, Differential- und Integralgleichungen der Mechanik und Physik, Vol. 2, Vieweg, Braunschweig 1935.
[16] S. Banach, Mechanics (in Polish), Vol. 2, Czytelnik, Warszawa 1947.
[17] K. L. Lang, Astrophysical Data: Planets and Stars, Springer, New York 1991.
[18] W. Dittrich and M. Reuter, Classical and Quantum Dynamics, Springer, Berlin 1992.